JOURNAL OF MULTIVARIATE ANALYSIS 5, 415–424 (1975)

Decomposition of Multivariate Infinitely Divisible Characteristic Functions with Absolutely Continuous Poisson Spectral Measures

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Communicated by E. Lukacs

We shall consider the decomposition problem of multivariate infinitely divisible characteristic functions which have no Gaussian component and have absolutely continuous Poisson spectral measures. Under the condition that $A = \{x; f(x) > 0\}$ is open, where f is the density of spectral measure, we shall show that a known sufficient condition for the membership of the class I_{0m} (i.e., infinitely divisible characteristic functions having only infinitely divisible factors) is also necessary.

1. INTRODUCTION

In this paper we shall treat the decomposition problem of multivariate infinitely divisible characteristic functions (abbreviated i.d.c.f.) which have no Gaussian components and have absolutely continuous Poisson spectral measures in their Lévy–Khintchine canonical representations. It is well-known that an m-dimensional i.d.c.f. $\phi(t)$ has the following representation.

$$\phi(t) = \exp\left[i(\beta, t) - Q(t) + \int_{R^m-\{0\}} K(t, x)\, d\nu(x)\right], \tag{1.1}$$

$$K(t, x) = e^{i(t,x)} - 1 - i(t, x)/(1 + |x|^2),$$

$$t = (t_1, t_2, \ldots, t_m) \in R^m,$$

$$Q(t) = \sum_{j,k=1}^{m} \gamma_{jk} t_j t_k,$$

Received December 9, 1974; revised March 12, 1975.

AMS 1970 subject classifications; Primary 42A96.

Key words and phrases: Decomposition, multivariate laws, infinitely divisible, absolutely continuous Poisson spectral measure, indecomposable factor.

where $\beta \in R^m$ is a constant, $Q(t)$ is a nonnegative definite quadratic form (called the Gaussian component of the representation (1, 1)), (t, x) is the natural inner product of R^m, $|x|^2 = (x, x)$ and v is a nonnegative measure on $R^m - \{0\}$ (called the Poisson spectral measure of the representation (1, 1)) with the condition

$$\int_{R^m - \{0\}} |x|^2/(1 + |x|^2)\, dv(x) < +\infty.$$

We shall be interested in the problem of characterizing those m-dimensional i.d.c.f.'s which have only i.d. factors, i.e., if $\phi(t)$ is decomposed into the product of two other c.f.'s $\phi(t) = \phi_1(t)\,\phi_2(t)$, then both ϕ_1 and ϕ_2 are i.d. This class of i.d.c.f.'s is usually referred to as the class I_{0m}. Many authors have studied this class and obtained various important results. Nevertheless, no complete description of this class has as yet been obtained in either case $m = 1$ or case $m \geqslant 2$. It is well known that the problem has completely different aspects according to whether Gaussian components exist or not in Lévy–Khintchine representations.

In a previous work [4], the present author considered this problem in the case $m = 1$ and $Q = 0$ and v is absolutely continuous. Using a method of Cramér, which has been elaborated by Cuppens [2], the author showed that the sufficient condition for this problem due to Ostrovskii is also necessary if $m = 1$. And the purpose of this paper is to extend this result to multidimensional case. Because of difficulties caused by multidimensionality the obtained result is not complete, but it shows that the problem is connected with natures of supports of spectral measures but not with spectral measure itself.

2. Preliminaries and Statement of Results

We shall list several notations used in the sequel. Let $A \subset R^m$. A^* means the convex hull of A. A^c means the closure of A. $(k)A$, $k \geqslant 1$, means the kth iterated vectorial sum of A defined as

$$(1)A = A, \qquad (k+1)A = (k)A + A, \qquad k \geqslant 1,$$

where the symbol "+" means the vectorial summation of two sets of R^m.

It is obvious that A^* and $(k)A$ are open if A is open. A is said to be contained in a half-space (through the origin) if the interior of A^* does not contain the origin. This is the same as saying that $A \subset \{x; (x, y) \geqslant 0\}$ for some $y \neq 0, \in R^m$.

If g and h are Borel measurable functions defined on R^m, $g * h$ means their convolution. The k-th iterated convolution of a Borel measurable function g_1 are denoted by g_k, $k \geqslant 1$, and are defined recurrently as

$$g_{k+1} = g_k * g_1, \qquad k \geqslant 1.$$

Q_ϵ means the set $\{x; \mid x \mid \leqslant \epsilon\}$. S means the unit sphere of R^m. $H(i; l, n,...)$ is the constant $i!/(l!\, n!...)$.

Let $\phi(t)$ be an m-dimensional i.d.c.f. which has no Gaussian component and has an absolutely continuous Poisson spectral measure. ϕ has the following representation.

$$\phi(t) = \exp\left[i(\beta, t) + \int_{R^m} K(t, x) f(x)\, dx\right], \tag{2.1}$$

where f is the density function of the Poisson spectral measure. For such an i.d.c.f. we shall prove the following results.

THEOREM. *Suppose the set $A = \{x; f(x) > 0\}$ is open. If $\phi \in I_{0m}$, then*

$$A^* \cap \left(\bigcup_{k=2}^{\infty} (k)A\right) = \varnothing. \tag{2.2}$$

COROLLARY 1. *Suppose $A = \{x; f(x) > 0\}$ is open and bounded. Then condition* (2.2) *is necessary and sufficient for ϕ to belong to I_{0m}.*

COROLLARY 2. *Suppose $A = \{x; f(x) > 0\}$ is open and f is integrable on A. Then condition* (2.2) *is necessary and sufficient for ϕ to belong to I_{0m}.*

COROLLARY 3. *Suppose $\phi \in I_{0m}$. Then the interior of the set $A = \{x; f(x) > 0\}$ is contained in a half-space.*

In the next section we shall prove these results after preparing several results.

Remark 1. If $m = 1$, condition (2.2) implies that A is contained in an interval of the form $[c, 2c]$ or $[-2c, -c]$, $c > 0$, and this is a necessary and sufficient condition for ϕ to belong to I_{01} without any restrictions on f, (see [5]).

Remark 2. The assumption that $A = \{x; f(x) > 0\}$ is open is trivially satisfied if f is continuous. Also, if f is a.e. continuous on R^m, we can assume that A is open. In general, if the closure of the interior of the set A contains A, the boundary set of A is of Lebesgue measure 0 and we can suppose that A is open from the first.

Remark 3. Under the assumption $\mathrm{Inf}_{x\in A} f(x) > 0$, an anologous result to our theorem is known [4, Chap. 6, Theorem 6.7.3, p. 289]. But condition (2.2) is replaced by an artificial condition (i.e., condition (K) in their book, which is the same as to assume the conclusion of Lemma 7 in the next section; see [4, Chap. 6, p. 287]). Our theorem shows that the assumption $\mathrm{Inf}_{x\in A} f(x) > 0$ is superfluous and condition (K) is a consequence of condition (2.2). Also, Cuppens [3] studied the case where f is a.e. continuous, extending his earlier result [2].

3. Auxiliary Lemmas and Proof of Theorem

Lemma 1. *Let a set $A \subset R^m$ be open. If A is not contained in any half-spaces, then $0 \in (n)A$ for all large n.*

Proof. We can take a finite set $Z = \{x_1, ..., x_r\} \subset A$ which is not contained in any half-spaces. Fix a small number $\epsilon > 0$ so that

$$U_i \equiv x_i + Q_\epsilon \subset A, \qquad i = 1, 2, ..., r.$$

Let $U = \bigcup_{i=1}^r U_i$; then

$$(n)U = \bigcup_{j_1+\cdots+j_r=n} [(j_1x_1 + \cdots + j_rx_r) + Q_{n\epsilon}].$$

Since $0 \in Z^*$, there are numbers $s_1, ..., s_r \geqslant 0$, $s_1 + \cdots + s_r = 1$ such that $0 = \sum_{i=1}^r s_ix_i$. Approximating $s_1, ..., s_r$ by rational numbers of the form $s_1/n, ..., s_r/n$, we can make $| j_1x_1 + \cdots + j_rx_r | < n\epsilon$ for all large n, where $j_1, ..., j_r$, $j_1 + \cdots + j_r = n$ are nonnegative integers. Therefore,

$$0 \in (j_1x_1 + \cdots + j_rx_r) + Q_{n\epsilon}$$

and $0 \in (n)U \subset (n)A$.

Lemma 2. *Let $A \subset R^m$ be bounded. Suppose $(q)A \supset Q_\epsilon$ for a certain q and $\epsilon > 0$. Then there is a positive number δ such that $(n)A \supset Q_{n\delta}$ for all large n.*

Proof. Fix a large number $K > 0$ such that $A \subset Q_K$. For every k, n such that $0 \leqslant n < q$, $k\epsilon > nK$,

$$(kq + n)A = (kq)A + (n)A \supset Q_{k\epsilon} + (n)A \supset Q_{k\epsilon - nK}.$$

Therefore, if we take $\delta = \epsilon/(2q)$ and $K > (2qK + \epsilon)/\epsilon$, then

$$k\epsilon - qK > (k + 1)\, q\delta > (kq + n)\delta,$$

and

$$(kq + n)A \supset Q_{k\epsilon - nK} \supset Q_{(kq+n)\delta}.$$

Lemma 3. *Let $Z = \{x_1, ..., x_r\} \subset R^m$ be a finite set. If Z is not contained in any half-spaces, then $Z + Q_K \supset Q_K$ for all large $K > 0$.*

Proof. Fix a number $\alpha > 0$ so that $Q_{2\alpha} \subset Z^*$. It is easy to see that, for all large $K > 0$, the closed segment with endpoints 0 and Ke is contained in $Z + Q_K$ if $(x, e) > \alpha$, $x \in Z$, $e \in S$. Fix such K. Suppose that there exists a point $e \in S$ such that $(x, e) \leqslant \alpha$ for every $x \in Z$. Then there are numbers $\lambda_1, \ldots, \lambda_r \geqslant 0$, $\lambda_1 + \cdots + \lambda_r = 1$ such that $2\alpha e = \sum_{k=1}^{r} \lambda_k x_k$. Then

$$2\alpha = (2\alpha e, e) = \sum_{k=1}^{r} \lambda_k(e, x_k) \leqslant \sum_{k=1}^{r} \lambda_k \alpha = \alpha.$$

This is a contradiction and, therefore, $Z + Q_K \supset Q_K$.

LEMMA 4. *Let $Z \subset R^m$ be a finite set which is not contained in any half-spaces. If we set $A = Z + Q_\epsilon$ with a positive constant ϵ, then $(n + 1)A \supset (n)A + Q_\epsilon$ for all large n.*

Proof. By Lemma 3, $Z + Q_{(n+1)\epsilon} \subset Q_{(n+1)\epsilon}$ for all large n. Then

$$(n + 1)A = (n)Z + [Z + Q_{(n+1)\epsilon}] \supset (n)Z + Q_{(n+1)\epsilon} = (n)A + Q_\epsilon .$$

LEMMA 5. *Let $A \subset R^m$ be bounded. Suppose that there exist integers k, r and positive numbers ϵ, γ such that*

$$(k + 1)A \supset (k)A + Q_\epsilon, \qquad (r)A \supset Q_\gamma .$$

Then we can find a positive number δ so that $(n)A = (n)(A \cup Q_\delta)$ for n large enough.

Proof. From the assumption and Lemma 2, there is an integer $q > 0$ and a positive number $\delta > 0$ such that $(n + 1)A \supset (n)A + Q_{2\delta}$, $(n)A \supset Q_{2n\delta}$, for all $n \geqslant q$. Fix a large number K such that $A \subset Q_K$. Take two nonnegative integers j, k, $j + k = n$. If $j \geqslant q$,

$$(n)A \supset (n - 1)A + Q_\delta \supset \cdots \supset (j)A + Q_{k\delta}$$

and if $j < q$,

$$(j)A + Q_{k\delta} \subset Q_{jK+n\delta} \subset Q_{qK+n\delta} .$$

Therefore, if $n > qK/\delta$ (i.e., $qK + n\delta < 2n\delta$) and $n \geqslant q$,

$$(j)A + Q_{k\delta} \subset Q_{qK+n\delta} \subset Q_{2n\delta} \subset (n)A.$$

Then

$$(n)A \supset \bigcup_{j+k=n} [(j)A + Q_{k\delta}] = (n)(A \cup Q_\delta);$$

that is, $(n)A = (n)(A \cup Q_\delta)$.

LEMMA 6. *Let $A \subset R^m$ be an open set with property $A^* \cap (q)A \neq \varnothing$ for a certain $q \geqslant 2$. We can find two nonvoid open sets B, C such that*

$$B \subset A, \qquad B \cap C = \varnothing, \qquad C \subset (q)B,$$
$$(n)(B \cup C) = (n)B \qquad \textit{for all large } n.$$

Proof. Let $z \in A^* \cup (q)A$. We can represent z as

$$z = \sum_{k=0}^{r} s_k x_k = y_1 + \cdots + y_q,$$

where $Z = \{x_0, ..., x_r, y_1, ..., y_q\} \subset A$ and $s_0, ..., s_r \geqslant 0$, $s_0 + \cdots + s_r = 1$. We can suppose that $z \notin Z$ and z belongs to the interior of Z^*, changing each point of Z in their small neighborhoods if necessary. Set $X = -z + Z$. X is not contained in any half-spaces and $0 \notin X$. Fix a small number $\epsilon > 0$ so that $Q_\epsilon \subset X^*$, $(X + Q_\epsilon) \cap Q_\epsilon = \varnothing$ and $Z + Q_\epsilon \subset A$. From Lemmas 1, 4, and 5, there is a positive number $\delta < \epsilon$ such that, for all large n,

$$(n)(X + Q_\epsilon) = (n)[(X + Q_\epsilon) \cup Q_\delta].$$

To prove the lemma, it suffices to take $B = Z + Q_\epsilon$ and $C = z + Q_\epsilon$.

LEMMA 7. *Let $f = f_1$ be a nonnegative, bounded Borel measurable function defined on R^m and $A \equiv \{x; f(x) > 0\}$ be open and bounded. Suppose that there exist integers $q, r \geqslant 2$ and nonvoid open sets B, C such that*

$$A \cap C = \varnothing, \qquad B^c \subset C \cap (q)A,$$
$$(n)(A \cup C) = (n)A \qquad \textit{for all} \quad n \geqslant r.$$

Define the function $g_1(x) = g_1(x; \epsilon) = f_1(x) - \epsilon f_q(x)\, \chi_B(x)$, where $\chi_B(x)$ is the indicator function of the set B. Then, for all $\epsilon > 0$ small enough,

$$\sum_{n=1}^{\infty} \frac{1}{n!} g_n(x) \geqslant 0 \textit{ everywhere.}$$

Proof. Set $D_n = \bigcup_{k=1}^{n} [(n-k)A + (k)B]$, $n = 1, 2, ...$. We note that, for all $n \geqslant r$,

$$\begin{aligned} D_n{}^c &\subset [(n-1)(A \cup C)]^c + B^c \subset [(n-1)(A \cup C)]^c + C \\ &= (n-1)(A \cup C) + C \subset (n)A. \end{aligned}$$

Define the functions $d_1(x) = \epsilon f_q(x)\, \chi_B(x)$ and $h_1(x) = f_q(x) - d_1(x)$. g_n can be developed as

$$\begin{aligned} g_n &= \sum_{j+k=n} H(n; j, k)(-\epsilon)^k f_j * d_k \\ &= \sum_{j+2k=n} H(n; j, 2k)\, \epsilon^{2k} f_j * d_{2k} - \sum_{j+2k+1=n} H(n; j, 2k+1)\, \epsilon^{2k+1} f_j * d_{2k+1} \\ &\equiv g_n^{(+)} - g_n^{(-)}. \end{aligned}$$

It is easy to see that

$$\begin{gathered} g_n(x) \geqslant f_n(x) - g_n^{(-)}(x), \\ \lim_{\epsilon\to+0} \operatorname*{Max}_{x\in R^m} |g_n^{(-)}(x; \epsilon)| = 0. \end{gathered} \tag{3.1}$$

And, for all $n \geqslant r$,

$$\operatorname{Supp}(g_n^{(-)}) \subset D_n{}^c \subset (n)A = \{x; f_n(x) > 0\}.$$

Since f_n is continuous on R^m (from a well-known theorem of Lebesgue) if $n > 1$, it follows that $\inf_{x\in D_n{}^c} f_n(x) > 0$. Hence, from (3.1), if we take $\epsilon > 0$ small enough,

$$g_n(x) \geqslant 0 \text{ everywhere for } n = r, r+1,..., 2r-1.$$

For such ϵ, it follows also that $g_n(x) \geqslant 0$ everywhere for all $n \geqslant r$. For example, $g_{2r} = g_r * g_r \geqslant 0$.

Next note that, for $n = jq + s$, $0 \leqslant j$, $0 \leqslant s < q$,

$$\begin{aligned} f_{n+q} * d_{2k} &= f_s * (d_1 + h_1)^{*(j+1)} * d_{2k} \\ &\geqslant \sum_{\nu+\mu=j} H(j+1; \nu+1, \mu) f_s * d_{2k+\nu+1} * h_\mu\,, \end{aligned} \tag{3.2}$$

$$\begin{aligned} f_n * d_{2k+1} &= f_s * (d_1 + h_1)^{*j} * d_{2k+1} \\ &= \sum_{\nu+\mu=j} H(j; \nu, \mu) f_s * d_{2k+\nu+1} * h_\mu\,. \end{aligned} \tag{3.3}$$

Substituting (3.2) and (3.3) and neglecting some of positive terms, we get, for every $\alpha > 0$,

$$\alpha g_{n+q}^{(+)} - g_{n+1}^{(-)} \geqslant \sum_{jq+s+2k=n}^{s<q} \epsilon^{2k} \sum_{\nu+\mu=j} J(\alpha, \epsilon; n, j, k, q, \nu, \mu) f_s * d_{2k+\nu+1} * h_\mu\,,$$

$$\begin{aligned} J(\alpha, \epsilon; n, j, k, q, \nu, \mu) &= \alpha H(n+q; j+q, 2k)\, H(j+1; \nu+1, \mu) \\ &\quad - \epsilon H(n+1; j, 2k+1)\, H(j; \nu, \mu). \end{aligned}$$

Hence the left-hand side of this inequality is nonnegative everywhere for each $\alpha > 0$ if $\epsilon > 0$ is small enough. Finally,

$$\sum_{k=1}^{r+q-1} \frac{1}{k!} g_k \geqslant \sum_{k=1}^{r} \left[\frac{1}{2} \frac{1}{(k+q-1)!} g_{k+q-1}^{(+)} - \frac{1}{k!} g_k^{(-)}\right]$$

$$+ \sum_{k=r+1}^{r+q-1} \left[\frac{1}{2} \frac{1}{k!} g_k^{(+)} - \frac{1}{k!} g_k^{(-)}\right].$$

We have shown above that each of the summands in the first and second terms of the right-hand side of this inequality can be made nonnegative for $\epsilon > 0$ small enough.

Summing up all the arguments, we can conclude that

$$\sum_{k=1}^{\infty} \frac{1}{k!} g_k = \sum_{k=1}^{r+q-1} \frac{1}{k!} g_k + \sum_{k=r+q}^{\infty} \frac{1}{k!} g_k$$

is nonnegative everywhere if $\epsilon > 0$ is small enough.

LEMMA 8. *Let $g = g_1$ be a bounded, integrable function defined on R^m with properties*

$$\begin{gathered} g(x) < 0 \text{ with positive Lebesgue measure,} \\ g_\infty(x) \equiv \sum_{n=1}^{\infty} \frac{1}{n!} g_n(x) \geqslant 0 \text{ everywhere.} \end{gathered} \tag{3.4}$$

Define the function

$$\psi(t) = \exp\left[\int_{R^m} K(t, x) g(x)\, dx\right].$$

Then $\psi(t)$ is a c.f. which is not i.d.

Proof. Define the constant $\gamma = (\gamma_1, \ldots, \gamma_m) \in R^m$ and α by

$$\gamma_k = \int_{R^m} x_k g(x)/(1 + |x|^2)\, dx, \qquad k = 1, \ldots, m,$$

$$\alpha = \int_{R^m} g(x)/(1 + |x|^2)\, dx.$$

Then

$$\begin{aligned}\psi(t) &= e^{-\alpha-i(\gamma,t)} \exp\left[\int_{R^m} e^{i(x,t)} g(x)\, dx\right] \\ &= e^{-\alpha-i(\gamma,t)}\left[1+\sum_{n=1}^{\infty}\frac{1}{n!}\left\{\int_{R^m} e^{i(x,t)}g(x)\,dx\right\}^n\right] \\ &= e^{-\alpha-i(\gamma,t)}\left[1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{R^m} e^{i(x,t)}g_n(x)\,dx\right].\end{aligned}$$

Therefore, $\psi(t)$ is the Fourier transform of the positive measure

$$dP(x) = e^{-\alpha}[d\epsilon(x+\gamma) + g_\infty(x+\gamma)\,dx],$$

where $d\epsilon$ is the unit mass at the origin. Since $\psi(0) = 1$, dP is a probability measure and ψ is its c.f.

The fact that ψ cannot be i.d. follows from the uniqueness of the representation of functions which can be represented in form (1.1) with some signed measure v.

Now we can begin the proofs of Theorem and Corollaries.

Proof of Theorem. Suppose that condition (2.2) is not satisfied. From Lemmas 6 and 7, there exists a bounded integrable function $g = g_1$ which satisfies conditions (3.4) and $f(x) - g(x) \geqslant 0$ everywhere. Then

$$\begin{aligned}\psi(t) &= \exp\left[i(\beta,t) + \int_{R^m} K(t,x) f(x)\,dx\right] \\ &= \exp\left[i(\beta,t) + \int_{R^m} K(t,x)\{f(x)-g(x)\}\,dx\right] \exp\left[\int_{R^m} K(t,x)\, g(x)\,dx\right] \\ &\equiv \psi_1(t)\,\psi_2(t).\end{aligned}$$

ψ_1 is obviously an i.d.c.f. And, from Lemma 8, ψ_2 is also a c.f. which is not i.d. Therefore, $\psi \notin I_{0m}$.

Proof of Corollary 1. If $A = \{x; f(x) > 0\}$ is open and bounded, it is known that condition (2.2) assures that $\phi \in I_{0m}$ (see [4, Chap. 6, Theorem 6.6.1, p. 275]).

Proof of Corollary 2. It is enough to prove the sufficient part. Let ϕ be an i.d.c.f. defined by (1.1) with $Q = 0$. Let the spectral measure v be concentrated on an open set A and be of bounded variation. Fix a particular decomposition of ϕ, $\phi = \phi_1\phi_2$. A theorem due to Cuppens [3, Theorem 3, p. 125], then, tells us that ϕ_j, $j = 1, 2$, has the representation

$$\phi_j(t) = \exp\left[i(\beta_j, t) + \int_{R^m - \{0\}} K(t, x)\, dv_j(x)\right],$$

where $\beta_j \in R^m$ is a constant and v_j is a not necessarily nonnegative measure of bounded variation which is concentrated on $(A^*)^c \cap (\bigcup_{k=1}^{\infty} (k)A)$. If A satisfies condition (2.2), then $(A^*)^c \cap (\bigcup_{k=2}^{\infty} (k)A) = \varnothing$, $0 \notin A$, and, therefore, v_j is concentrated on A. From the proof of Lemma 8, it is seen that the probability measure P_j corresponding to ϕ_j has the form

$$dP_j(x) = C_j \left[d\epsilon(x - \beta_j) + \sum_{k=1}^{\infty} \frac{1}{k!}\, dv_j^{*k}(x - \beta_j)\right].$$

Since $A^* \cap ((k)A)^c = \varnothing$, $k \geqslant 2$, $v_j(E) = C_j^{-1} P_j(E + \beta_j) \geqslant 0$ for every $E \subset A$. Therefore v_j is nonnegative and ϕ_j is i.d. This completes the proof.

Proof of Corollary 3. If the interior of $A = \{x; f(x) > 0\}$ is not contained in any half-spaces, it follows from Lemma 1 that $0 \in (\text{Int } A)^* \cap (n)(\text{Int } A)$ for all large n. Therefore, condition (2.2) is violated.

Acknowledgments

The author sincerely express his thanks to Professor M. Huzii for his encouragement and advice. The author is also very much indebted to the referee for careful reading of the manuscript and bringing the results of Cuppens to his attention.

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